Exact Jacobian Elliptic Function Solutions to the Double Sine-Gordon Equation

Zuntao Fu[a,b], Shikuo Liu[a], and Shida Liu[a,b]

[a] School of Physics, Peking University, Beijing, 100871, China
[b] LTCS, Peking University, Beijing, 100871, China

Reprint requests to Z. F.; E-mail: fuzt@pku.edu.cn

Z. Naturforsch. **60a,** 301 – 312 (2005); received January 26, 2005

In this paper, two transformations are introduced to solve the double sine-Gordon equation by using knowledge of the elliptic equation and Jacobian elliptic functions. It is shown that different transformations can be applied to obtain more kinds of solutions to the double sine-Gordon equation.
PACS: 03.65.Ge

Key words: Jacobian Elliptic Function; Transformation; Sine-Gordon Equation.

1. Introduction

Sine-Gordon-type equations, including the single sine-Gordon (SSG) equation

$$u_{xt} = \alpha \sin u, \tag{1}$$

the double sine-Gordon (DSG) equation

$$u_{xt} = \alpha \sin u + \beta \sin 2u \tag{2}$$

and the triple sine-Gordon (TSG) equation

$$u_{xt} = \alpha \sin u + \beta \sin 2u + \gamma \sin 3u, \tag{3}$$

are widely applied in physics and engineering. For example, the DSG equation is a frequent object of study in numerous physical applications, such as Josephson arrays, ferromagnetic materials, charge density waves, smectic liquid crystal dynamics [1 – 5]. Actually, the SSG equation and the DSG equation also arise in nonlinear optics, ^{3}He spin waves and other fields. In a resonant fivefold degenerate medium, the propagation and creation of ultra-short optical pulses, the SSG and the DSG models are usually used. However, in some cases, one has to consider other sine-Gordon equations. For instance, the TSG equation is used to describe the propagation of strictly resonant sharp line optical pulses [6].

Due to the wide applications of sine-Gordon-type equations, many solutions to them in different functional forms, such as $\tan^{-1}\coth\xi$, $\tan^{-1}\tanh\xi$, $\tan^{-1}\operatorname{sech}\xi$, $\tan^{-1}\operatorname{sn}\xi$, have been obtained by different methods [7 – 12]. Due to the special forms of sine-Gordon-type equations, it is rather difficult to solve them directly, so there is need for some appropriate transformations. In this paper, based on the introduced transformations, we will show systematical results about solutions for the DSG equation (2) by using knowledge of elliptic equation and Jacobian elliptic functions [13 – 19].

2. The First Kind of Transformation and Solutions to the DSG Equation

In order to solve sine-Gordon-type equations, certain transformations must be introduced. For example, the transformation

$$u = 2\tan^{-1} v \quad \text{or} \quad v = \tan\frac{u}{2}, \tag{4}$$

has been introduced in [7, 9] to solve the DSG equation.

When the transformation (4) is considered, we have

$$\sin u = \frac{2\tan\frac{u}{2}}{1+\tan^2\frac{u}{2}} = \frac{2v}{1+v^2}, \tag{5}$$

and

$$u_{tx} = \frac{2}{1+v^2} v_{tx} - \frac{4v}{(1+v^2)^2} v_t v_x. \tag{6}$$

Combining (5) and (6) with (2), the DSG equation can be rewritten as

$$(1+v^2)v_{tx} - 2v v_t v_x - (\alpha+2\beta)v - (\alpha-2\beta)v^3 = 0, \tag{7}$$

which can be solved in the frame

$$v = v(\xi), \quad \xi = k(x - ct), \tag{8}$$

where k and c are wave number and wave speed, respectively.

Substituting (8) into (7), we have

$$(1+v^2)\frac{\mathrm{d}^2 v}{\mathrm{d}\xi^2} - 2v(\frac{\mathrm{d}v}{\mathrm{d}\xi})^2 + \alpha_1 v + \beta v^3 = 0, \tag{9}$$

with

$$\alpha_1 = \frac{\alpha + 2\beta}{k^2 c}, \quad \beta_1 = \frac{\alpha - 2\beta}{k^2 c}. \tag{10}$$

And then we suppose (9) has the following solution

$$v = v(y) = \sum_{j=0}^{j=n} b_j y^j, \quad b_n \neq 0, \quad y = y(\xi), \tag{11}$$

where y satisfies the elliptic equation [13–17, 20]

$$y'^2 = a_0 + a_2 y^2 + a_4 y^4, \quad a_4 \neq 0, \tag{12}$$

with $y' = \frac{\mathrm{d}y}{\mathrm{d}\xi}$, then

$$y'' = a_2 y + 2a_4 y^3. \tag{13}$$

The n in (11) can be determined by the partial balance between the highest order derivative terms and the highest degree nonlinear term in (9). Here we know that the degree of v is

$$O(v) = O(y^n) = n, \tag{14}$$

and from (12) and (13), one has

$$O(y'^2) = O(y^4) = 4, \quad O(y'') = O(y^3) = 3, \tag{15}$$

and actually one has

$$O(y^{(l)}) = l + 1. \tag{16}$$

So one has

$$\begin{aligned} &O(v) = n, \quad O(v') = n+1, \\ &O(v'') = n+2, \quad O(v^{(l)}) = n+l. \end{aligned} \tag{17}$$

For the DSG equation (2), we have $n = 1$, so the ansatz solution of (9) can be rewritten as

$$v = b_0 + b_1 y, \quad b_1 \neq 0. \tag{18}$$

Substituting (18) into (9) results in an algebraic equation for y, which can be used to determine expansion coefficients in (18) and some constraints can also be obtained. Here we have

$$b_0 = 0, \tag{19a}$$

$$b_1^2 = \frac{a_2 + \alpha_1}{2a_0}, \tag{19b}$$

$$b_1^2 = \frac{2a_4}{a_2 - \beta_1}. \tag{19c}$$

From (19b) and (19c), the constraints can be determined as

$$\frac{a_2 + \alpha_1}{2a_0} > 0, \quad \frac{2a_4}{a_2 - \beta_1} > 0, \tag{20}$$

and

$$(a_2^2 - 4a_0 a_4)k^4 c^2 + 4\beta a_2 k^2 c - (\alpha^2 - 4\beta^2) = 0, \tag{21}$$

with

$$4\beta^2 a_2^2 + (a_2^2 - 4a_0 a_4)(\alpha^2 - 4\beta^2) \geq 0. \tag{22}$$

It is worth noting that if $a_0 = 0$, then we have

$$b_0 = 0, \quad b_1^2 = \frac{2a_4}{a_2 - \beta_1}, \tag{23}$$

with the constraint

$$a_2 + \alpha_1 = 0, \quad \frac{2a_4}{a_2 - \beta_1} > 0. \tag{24}$$

Considering the constraints (20) and (22) or (23) and (24), the solutions to the elliptic equation (12) can be used to derive the final results. Here some cases can be obtained.

Case 1. If $a_0 = 0$, $a_2 = 1$, $a_4 = -1$, then

$$\begin{aligned} &y = \operatorname{sech}\xi, \quad b_0 = 0, \\ &b_1 = \pm\sqrt{-\frac{\alpha + 2\beta}{\alpha}}, \quad c = -\frac{\alpha + 2\beta}{k^2}, \end{aligned} \tag{25}$$

with constraint

$$\frac{\alpha + 2\beta}{\alpha} < 0, \tag{26}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_1 = 2\tan^{-1}(\pm\sqrt{-\frac{\alpha + 2\beta}{\alpha}}\operatorname{sech}\xi). \tag{27}$$

Case 2. If $a_0 = 0$, $a_2 = 1$, $a_4 = 1$, then

$$y = \operatorname{csch}\xi, \quad b_0 = 0, \quad b_1 = \pm\sqrt{\frac{\alpha+2\beta}{\alpha}}, \quad c = -\frac{\alpha+2\beta}{k^2}, \tag{28}$$

with constraint

$$\frac{\alpha+2\beta}{\alpha} > 0, \tag{29}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_2 = 2\tan^{-1}\Big(\pm\sqrt{\frac{\alpha+2\beta}{\alpha}}\operatorname{csch}\xi\Big). \tag{30}$$

Case 3. If $a_0 = 1$, $a_2 = -2$, $a_4 = 1$, then

$$y = \tanh\xi, \quad b_0 = 0, \quad b_1 = \pm\sqrt{\frac{2\beta+\alpha}{2\beta-\alpha}}, \quad c = -\frac{\alpha^2-4\beta^2}{8\beta k^2}, \tag{31}$$

with constraint

$$\alpha^2 < 4\beta^2, \tag{32}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_3 = 2\tan^{-1}\Big(\pm\sqrt{\frac{2\beta+\alpha}{2\beta-\alpha}}\tanh\xi\Big). \tag{33}$$

Case 4. If $a_0 = 1$, $a_2 = -2$, $a_4 = 1$, then

$$y = \coth\xi, \quad b_0 = 0, \quad b_1 = \pm\sqrt{\frac{2\beta+\alpha}{2\beta-\alpha}}, \quad c = -\frac{\alpha^2-4\beta^2}{8\beta k^2}, \tag{34}$$

with constraint

$$\alpha^2 < 4\beta^2, \tag{35}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_4 = 2\tan^{-1}\Big(\pm\sqrt{\frac{2\beta+\alpha}{2\beta-\alpha}}\coth\xi\Big). \tag{36}$$

Case 5. If $a_0 = \frac{1}{4}$, $a_2 = -\frac{1}{2}$, $a_4 = \frac{1}{4}$, then

$$y = \frac{\tanh\xi}{1\pm\operatorname{sech}\xi}, \quad b_0 = 0, \quad b_1 = \pm\sqrt{\frac{2\beta+\alpha}{2\beta-\alpha}}, \quad c = -\frac{\alpha^2-4\beta^2}{2\beta k^2}, \tag{37}$$

with constraint

$$\alpha^2 < 4\beta^2, \tag{38}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_5 = 2\tan^{-1}\Big(\pm\sqrt{\frac{2\beta+\alpha}{2\beta-\alpha}}\frac{\tanh\xi}{1\pm\operatorname{sech}\xi}\Big). \tag{39}$$

Case 6. If $a_0 = 1-m^2$, $a_2 = 2m^2-1$, $a_4 = -m^2$, where $0 \le m \le 1$ is called the modulus of Jacobian elliptic functions [21–23], then

$$y = \operatorname{cn}\xi, \quad b_0 = 0, \quad b_1 = \pm\sqrt{\frac{\alpha+2\beta+(2m^2-1)k^2c}{2(1-m^2)k^2c}}, \quad c = \frac{2(1-2m^2)\beta \pm \sqrt{\alpha^2-16\beta^2m^2(1-m^2)}}{k^2}, \tag{40}$$

with

$$\alpha^2 - 16\beta^2m^2(1-m^2) \ge 0, \quad 0 < m < 1, \tag{41}$$

where k is an arbitrary constant, and $\operatorname{cn}\xi$ is the Jacobian elliptic cosine function [20–23]. So the solution to the DSG equation (2) is

$$u_6 = 2\tan^{-1}\Big(\pm\sqrt{\frac{\alpha+2\beta+(2m^2-1)k^2c}{2(1-m^2)k^2c}}\operatorname{cn}\xi\Big). \tag{42}$$

Case 7. If $a_0 = -m^2$, $a_2 = 2m^2-1$, $a_4 = 1-m^2$, then

$$y = \operatorname{nc}\xi \equiv \frac{1}{\operatorname{cn}\xi}, \quad b_0 = 0, \quad b_1 = \pm\sqrt{-\frac{\alpha+2\beta+(2m^2-1)k^2c}{2m^2k^2c}}, \quad c = \frac{2(1-2m^2)\beta \pm \sqrt{\alpha^2-16\beta^2m^2(1-m^2)}}{k^2}, \tag{43}$$

with

$$\alpha^2 - 16\beta^2m^2(1-m^2) \ge 0, \quad 0 < m < 1, \tag{44}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_7 = 2\tan^{-1}\Big(\pm\sqrt{-\frac{\alpha+2\beta+(2m^2-1)k^2c}{2m^2k^2c}}\operatorname{nc}\xi\Big). \tag{45}$$

Case 8. If $a_0 = 1$, $a_2 = 2m^2 - 1$, $a_4 = (m^2-1)m^2$, then

$$y = \mathrm{sd}\,\xi \equiv \frac{\mathrm{sn}\xi}{\mathrm{dn}\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{\alpha + 2\beta + (2m^2-1)k^2c}{2k^2c}}, \tag{46}$$
$$c = \frac{2(1-2m^2)\beta \pm \sqrt{\alpha^2 - 16\beta^2 m^2(1-m^2)}}{k^2},$$

with

$$\alpha^2 - 16\beta^2 m^2(1-m^2) \geq 0, \quad 0 < m < 1, \tag{47}$$

where k is an arbitrary constant, and $\mathrm{sn}\xi$ is the Jacobian elliptic sine function and $\mathrm{dn}\xi$ is the Jacobian elliptic function of the third kind [20–23]. So the solution to the DSG equation (2) is

$$u_8 = 2\tan^{-1}(\pm\sqrt{\frac{\alpha + 2\beta + (2m^2-1)k^2c}{2k^2c}}\mathrm{sd}\,\xi). \tag{48}$$

Case 9. If $a_0 = (m^2-1)m^2$, $a_2 = 2m^2 - 1$, $a_4 = 1$, then

$$y = \mathrm{ds}\,\xi \equiv \frac{\mathrm{dn}\xi}{\mathrm{sn}\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{\alpha + 2\beta + (2m^2-1)k^2c}{2(m^2-1)m^2k^2c}}, \tag{49}$$
$$c = \frac{2(1-2m^2)\beta \pm \sqrt{\alpha^2 - 16\beta^2 m^2(1-m^2)}}{k^2},$$

with

$$\alpha^2 - 16\beta^2 m^2(1-m^2) \geq 0, \quad 0 < m < 1, \tag{50}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_9 = 2\tan^{-1}\Big(\pm\sqrt{\frac{\alpha + 2\beta + (2m^2-1)k^2c}{2(m^2-1)m^2k^2c}}\mathrm{ds}\,\xi\Big). \tag{51}$$

Case 10. If $a_0 = 1$, $a_2 = -(1+m^2)$, $a_4 = m^2$, then

$$y = \mathrm{sn}\xi, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{\alpha + 2\beta - (1+m^2)k^2c}{2k^2c}}, \tag{52}$$
$$c = \frac{2(1+m^2)\beta \pm \sqrt{(1-m^2)^2\alpha^2 + 16\beta^2 m^2}}{(1-m^2)^2k^2},$$

with

$$\frac{2\beta}{k^2c} > (1+m^2), \quad 0 < m < 1, \tag{53}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{10} = 2\tan^{-1}\Big(\pm\sqrt{\frac{\alpha + 2\beta - (1+m^2)k^2c}{2k^2c}}\mathrm{sn}\,\xi\Big). \tag{54}$$

Case 11. If $a_0 = 1$, $a_2 = -(1+m^2)$, $a_4 = m^2$, then

$$y = \mathrm{cd}\,\xi \equiv \frac{\mathrm{cn}\,\xi}{\mathrm{dn}\,\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{\alpha + 2\beta - (1+m^2)k^2c}{2k^2c}}, \tag{55}$$
$$c = \frac{2(1+m^2)\beta \pm \sqrt{(1-m^2)^2\alpha^2 + 16\beta^2 m^2}}{(1-m^2)^2k^2},$$

with

$$\frac{2\beta}{k^2c} > (1+m^2), \quad 0 < m < 1, \tag{56}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{11} = 2\tan^{-1}\Big(\pm\sqrt{\frac{\alpha + 2\beta - (1+m^2)k^2c}{2k^2c}}\mathrm{cd}\,\xi\Big). \tag{57}$$

Case 12. If $a_0 = m^2$, $a_2 = -(1+m^2)$, $a_4 = 1$, then

$$y = \mathrm{ns}\,\xi \equiv \frac{1}{\mathrm{sn}\,\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{\alpha + 2\beta - (1+m^2)k^2c}{2m^2k^2c}}, \tag{58}$$
$$c = \frac{2(1+m^2)\beta \pm \sqrt{(1-m^2)^2\alpha^2 + 16\beta^2 m^2}}{(1-m^2)^2k^2},$$

with

$$\frac{2\beta}{k^2c} > (1+m^2), \quad 0 < m < 1, \tag{59}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{12} = 2\tan^{-1}\Big(\pm\sqrt{\frac{\alpha + 2\beta - (1+m^2)k^2c}{2m^2k^2c}}\mathrm{ns}\,\xi\Big). \tag{60}$$

Case 13. If $a_0 = m^2$, $a_2 = -(1+m^2)$, $a_4 = 1$, then

$$y = \mathrm{dc}\,\xi \equiv \frac{\mathrm{dn}\,\xi}{\mathrm{cn}\,\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{\alpha+2\beta-(1+m^2)k^2c}{2m^2k^2c}}, \tag{61}$$
$$c = \frac{2(1+m^2)\beta \pm \sqrt{(1-m^2)^2\alpha^2+16\beta^2m^2}}{(1-m^2)^2k^2},$$

with

$$\frac{2\beta}{k^2c} > (1+m^2), \quad 0 < m < 1, \tag{62}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{13} = 2\tan^{-1}\left(\pm\sqrt{\frac{\alpha+2\beta-(1+m^2)k^2c}{2m^2k^2c}}\mathrm{dc}\,\xi\right). \tag{63}$$

Case 14. If $a_0 = -(1-m^2)$, $a_2 = 2-m^2$, $a_4 = -1$, then

$$y = \mathrm{dn}\,\xi, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{-\frac{\alpha+2\beta+(2-m^2)k^2c}{2(1-m^2)k^2c}}, \tag{64}$$
$$c = \frac{2(m^2-2)\beta \pm \sqrt{m^2\alpha^2+16\beta^2(1-m^2)}}{m^4k^2},$$

with

$$\frac{2\beta}{k^2c} + (2-m^2) < 0, \quad 0 < m < 1, \tag{65}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{14} = 2\tan^{-1}\left(\pm\sqrt{-\frac{\alpha+2\beta+(2-m^2)k^2c}{2(1-m^2)k^2c}}\mathrm{dn}\,\xi\right). \tag{66}$$

Case 15. If $a_0 = -1$, $a_2 = 2-m^2$, $a_4 = -(1-m^2)$, then

$$y = \mathrm{nd}\,\xi \equiv \frac{1}{\mathrm{dn}\,\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{-\frac{\alpha+2\beta+(2-m^2)k^2c}{2k^2c}}, \tag{67}$$
$$c = \frac{2(m^2-2)\beta \pm \sqrt{m^2\alpha^2+16\beta^2(1-m^2)}}{m^4k^2},$$

with

$$\frac{2\beta}{k^2c} + (2-m^2) < 0, \quad 0 < m < 1, \tag{68}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{15} = 2\tan^{-1}\left(\pm\sqrt{-\frac{\alpha+2\beta+(2-m^2)k^2c}{2k^2c}}\mathrm{nd}\,\xi\right). \tag{69}$$

Case 16. If $a_0 = 1$, $a_2 = 2-m^2$, $a_4 = 1-m^2$, then

$$y = \mathrm{sc}\xi \equiv \frac{\mathrm{sn}\xi}{\mathrm{cn}\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{\alpha+2\beta+(2-m^2)k^2c}{2k^2c}}, \tag{70}$$
$$c = \frac{2(m^2-2)\beta \pm \sqrt{m^2\alpha^2+16\beta^2(1-m^2)}}{m^4k^2},$$

with

$$\frac{2\beta}{k^2c} + (2-m^2) > 0, \quad 0 < m < 1, \tag{71}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{16} = 2\tan^{-1}\left(\pm\sqrt{\frac{\alpha+2\beta+(2-m^2)k^2c}{2k^2c}}\mathrm{sc}\,\xi\right). \tag{72}$$

Case 17. If $a_0 = 1-m^2$, $a_2 = 2-m^2$, $a_4 = 1$, then

$$y = \mathrm{cs}\xi \equiv \frac{\mathrm{cn}\,\xi}{\mathrm{sn}\,\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{\alpha+2\beta+(2-m^2)k^2c}{2(1-m^2)k^2c}}, \tag{73}$$
$$c = \frac{2(m^2-2)\beta \pm \sqrt{m^2\alpha^2+16\beta^2(1-m^2)}}{m^4k^2},$$

with

$$\frac{2\beta}{k^2c} + (2-m^2) > 0, \quad 0 < m < 1, \tag{74}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{17} = 2\tan^{-1}\left(\pm\sqrt{\frac{\alpha+2\beta+(2-m^2)k^2c}{2(1-m^2)k^2c}}\mathrm{cs}\,\xi\right). \tag{75}$$

Case 18. If $a_0 = \frac{1-m^2}{4}$, $a_2 = \frac{1+m^2}{2}$, $a_4 = \frac{1-m^2}{4}$, then

$$y = \frac{\mathrm{cn}\xi}{1 \pm \mathrm{sn}\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{2(\alpha+2\beta)+(1+m^2)k^2c}{(1-m^2)k^2c}}, \tag{76}$$
$$c = \frac{-(1+m^2)\beta \pm \sqrt{m^2\alpha^2+\beta^2(1-m^2)^2}}{m^2k^2},$$

with

$$\frac{4\beta}{k^2c} + (1+m^2) > 0, \quad 0 < m < 1, \tag{77}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{18} = 2\tan^{-1}\Big(\pm\sqrt{\frac{2(\alpha+2\beta)+(1+m^2)k^2c}{(1-m^2)k^2c}} \cdot \frac{\mathrm{cn}\,\xi}{1\pm\mathrm{sn}\,\xi}\Big). \tag{78}$$

Case 19. If $a_0 = -\frac{1-m^2}{4}$, $a_2 = -\frac{1+m^2}{2}$, $a_4 = \frac{1-m^2}{4}$, then

$$y = \frac{\mathrm{dn}\xi}{1 \pm m\mathrm{sn}\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{-\frac{2(\alpha+2\beta)+(1+m^2)k^2c}{(1-m^2)k^2c}}, \tag{79}$$
$$c = \frac{-(1+m^2)\beta \pm \sqrt{m^2\alpha^2+\beta^2(1-m^2)^2}}{m^2k^2},$$

with

$$\frac{4\beta}{k^2c} + (1+m^2) < 0, \quad 0 < m < 1, \tag{80}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{19} = 2\tan^{-1}\Big(\pm\sqrt{-\frac{2(\alpha+2\beta)+(1+m^2)k^2c}{(1-m^2)k^2c}} \cdot \frac{\mathrm{dn}\,\xi}{1\pm m\mathrm{sn}\,\xi}\Big). \tag{81}$$

Case 20. If $a_0 = \frac{m^2}{4}$, $a_2 = -\frac{2-m^2}{2}$, $a_4 = \frac{m^2}{4}$, then

$$y = \frac{m\mathrm{sn}\,\xi}{1 \pm \mathrm{dn}\,\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{2(\alpha+2\beta)-(2-m^2)k^2c}{m^2k^2c}}, \tag{82}$$
$$c = \frac{(2-m^2)\beta \pm \sqrt{(1-m^2)\alpha^2+\beta^2m^4}}{(1-m^2)k^2},$$

with

$$\frac{4\beta}{k^2c} > (2-m^2), \quad 0 < m < 1, \tag{83}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{20} = 2\tan^{-1}\Big(\pm\sqrt{\frac{2(\alpha+2\beta)-(2-m^2)k^2c}{m^2k^2c}} \cdot \frac{m\mathrm{sn}\,\xi}{1\pm\mathrm{dn}\,\xi}\Big). \tag{84}$$

Case 21. If $a_0 = \frac{1}{4}$, $a_2 = \frac{1-2m^2}{2}$, $a_4 = \frac{1}{4}$, then

$$y = \frac{\mathrm{sn}\,\xi}{1 \pm \mathrm{cn}\,\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{2(\alpha+2\beta)+(1-2m^2)k^2c}{k^2c}}, \tag{85}$$
$$c = \frac{(2m^2-1)\beta \pm \sqrt{m^2(m^2-1)\alpha^2+\beta^2}}{m^2(m^2-1)k^2},$$

with

$$m^2(m^2-1)\alpha^2+\beta^2 > 0, \quad 0 < m < 1, \tag{86}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{21} = 2\tan^{-1}\Big(\pm\sqrt{\frac{2(\alpha+2\beta)+(1-2m^2)k^2c}{k^2c}} \cdot \frac{\mathrm{sn}\,\xi}{1\pm\mathrm{cn}\,\xi}\Big). \tag{87}$$

Case 22. If $a_0 = \frac{1}{4}$, $a_2 = -\frac{2-m^2}{2}$, $a_4 = \frac{m^4}{4}$, then

$$y = \frac{\mathrm{sn}\,\xi}{1 \pm \mathrm{dn}\,\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{2(\alpha+2\beta)-(2-m^2)k^2c}{k^2c}}, \tag{88}$$
$$c = \frac{(2-m^2)\beta \pm \sqrt{(1-m^2)\alpha^2+\beta^2m^4}}{(1-m^2)k^2},$$

with

$$\frac{4\beta}{k^2c} > (2-m^2), \quad 0 < m < 1, \tag{89}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{22} = 2\tan^{-1}\Big(\pm\sqrt{\frac{2(\alpha+2\beta)-(2-m^2)k^2c}{k^2c}} \cdot \frac{\mathrm{sn}\,\xi}{1\pm\mathrm{dn}\,\xi}\Big). \tag{90}$$

Remark 1. The solutions u_{18} to u_{22} in terms of rational functions of elliptic functions have not been reported in the literature, they are new solutions to the DSG equation (2).

Remark 2. The solutions from u_6 to u_{22} in terms of Jacobian elliptic functions have not been given in [11].

Remark 3. In [10], Peng solved the DSG equation in form of

$$u_{xt} = \sin u + \lambda \sin 2u, \tag{91}$$

and obtained some solutions in terms of Jacobian elliptic functions. He pointed out he can obtain solutions to the DSG equation (2) with $\alpha = 1$ when $\lambda = 0$. However, his conclusion is wrong, for the coefficients of solutions obtained in terms of sn ((40) in [10]), dn ((46) in [10]), ns ((51) in [10]) and dc ((52) in [10]) are imaginary, whereas they should be real. For example, from the constraint (41) in [10]

$$(1-m^2)^2k^2\omega^2 - 4\lambda(1+m^2)k\omega + 4\lambda^2 - 1 = 0, \tag{92}$$

if $\lambda = 0$, then we have

$$(1-m^2)^2k^2\omega^2 - 1 = 0, \tag{93}$$

i. e.

$$k\omega = \pm\frac{1}{1-m^2}. \tag{94}$$

Substituting (94) into solution (40) in [10]

$$u = 2\arctan\Big[\pm\sqrt{\frac{-(1+m^2)k\omega + 2\lambda + 1}{2k\omega}} \cdot \mathrm{sn}(kx - \omega t)\Big], \tag{95}$$

we can show that the coefficient $\sqrt{\frac{-(1+m^2)k\omega+1}{2k\omega}}$ becomes

$$im \quad \text{or} \quad i \quad \text{with} \quad i \equiv \sqrt{-1}. \tag{96}$$

So the solutions given by Peng in terms of sn ((40) in [10]), dn ((46) in [10]), ns ((51) in [10]) and dc ((52) in [10]) are not real solutions. This is contrary to the origin of the DSG equation (2).

Remark 4. Based on the above results, we can see that when the auxiliary equation, such as the elliptic equation (12), is applied to solve nonlinear evolution equations, the constraints must be involved, otherwise, the obtained solutions may be trivial.

3. The Second Kind of Transformation and Solutions to the DSG Equation

The second transformation under consideration is

$$u = 2\tan^{-1}\left(\frac{1}{v}\right) \quad \text{or} \quad \frac{1}{v} = \tan\frac{u}{2}, \tag{97}$$

which has been introduced in [7] to solve the DSG equation.

When the transformation (97) is considered, there are

$$\sin u = \frac{2\tan\frac{u}{2}}{1+\tan^2\frac{u}{2}} = \frac{2v}{1+v^2}, \tag{98}$$

and

$$u_{tx} = -\frac{2}{1+v^2}v_{tx} + \frac{4v}{(1+v^2)^2}v_t v_x. \tag{99}$$

Combining (98) and (99) with (2), the DSG equation can be rewritten as

$$(1+v^2)v_{tx} - 2vv_tv_x - (-\alpha+2\beta)v + (\alpha+2\beta)v^3 = 0. \tag{100}$$

We can see that the difference between (7) and (100) is that the $-\alpha$ in (7) is replaced by α in (100), so the solutions to (2) under the transformation (97) can be easily obtained by replacing α by $-\alpha$ and v by $\frac{1}{v}$ in solutions from u_1 to u_{22}.

Case 1. If $a_0 = 0$, $a_2 = 1$, $a_4 = -1$, then

$$y = \mathrm{sech}\,\xi, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{-\alpha+2\beta}{\alpha}}, c = -\frac{-\alpha+2\beta}{k^2}, \tag{101}$$

with constraint

$$\frac{-\alpha+2\beta}{\alpha} > 0, \tag{102}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{23} = 2\tan^{-1}\Big(\pm\sqrt{\frac{\alpha}{-\alpha+2\beta}}\cosh\xi\Big). \tag{103}$$

Case 2. If $a_0 = 0$, $a_2 = 1$, $a_4 = 1$, then

$$y = \operatorname{csch}\xi, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{-\frac{-\alpha+2\beta}{\alpha}}, \quad c = -\frac{-\alpha+2\beta}{k^2}, \tag{104}$$

with constraint

$$\frac{-\alpha+2\beta}{\alpha} < 0, \tag{105}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{24} = 2\tan^{-1}\Big(\pm\sqrt{-\frac{\alpha}{-\alpha+2\beta}}\sinh\xi\Big). \tag{106}$$

Case 3. If $a_0 = 1$, $a_2 = -2$, $a_4 = 1$, then

$$y = \tanh\xi, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{2\beta-\alpha}{2\beta+\alpha}}, \quad c = -\frac{\alpha^2-4\beta^2}{8\beta k^2}, \tag{107}$$

with constraint

$$\alpha^2 < 4\beta^2, \tag{108}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) recovers u_4.

Case 4. If $a_0 = 1$, $a_2 = -2$, $a_4 = 1$, then

$$y = \coth\xi, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{2\beta-\alpha}{2\beta+\alpha}}, \quad c = -\frac{\alpha^2-4\beta^2}{8\beta k^2}, \tag{109}$$

with constraint

$$\alpha^2 < 4\beta^2, \tag{110}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) recovers u_3.

Case 5. If $a_0 = \frac{1}{4}$, $a_2 = -\frac{1}{2}$, $a_4 = \frac{1}{4}$, then

$$y = \frac{\tanh\xi}{1\pm\operatorname{sech}\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{2\beta-\alpha}{2\beta+\alpha}}, \quad c = -\frac{\alpha^2-4\beta^2}{2\beta k^2}, \tag{111}$$

with constraint

$$\alpha^2 < 4\beta^2, \tag{112}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{25} = 2\tan^{-1}\Big(\pm\sqrt{\frac{2\beta+\alpha}{2\beta-\alpha}}\frac{1\pm\operatorname{sech}\xi}{\tanh\xi}\Big). \tag{113}$$

Case 6. If $a_0 = 1-m^2$, $a_2 = 2m^2-1$, $a_4 = -m^2$, then

$$y = \operatorname{cn}\xi, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{-\alpha+2\beta+(2m^2-1)k^2c}{2(1-m^2)k^2c}}, \tag{114}$$
$$c = \frac{2(1-2m^2)\beta \pm \sqrt{\alpha^2-16\beta^2m^2(1-m^2)}}{k^2},$$

with

$$\alpha^2-16\beta^2m^2(1-m^2) \geq 0, \quad 0 < m < 1, \tag{115}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{26} = 2\tan^{-1}\Big(\pm\sqrt{\frac{2(1-m^2)k^2c}{-\alpha+2\beta+(2m^2-1)k^2c}}\operatorname{nc}\xi\Big). \tag{116}$$

Case 7. If $a_0 = -m^2$, $a_2 = 2m^2-1$, $a_4 = 1-m^2$, then

$$y = \operatorname{nc}\xi, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{-\frac{-\alpha+2\beta+(2m^2-1)k^2c}{2m^2k^2c}}, \tag{117}$$
$$c = \frac{2(1-2m^2)\beta \pm \sqrt{\alpha^2-16\beta^2m^2(1-m^2)}}{k^2},$$

with

$$\alpha^2-16\beta^2m^2(1-m^2) \geq 0, \quad 0 < m < 1, \tag{118}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{27} = 2\tan^{-1}\Big(\pm\sqrt{-\frac{2m^2k^2c}{-\alpha+2\beta+(2m^2-1)k^2c}}\operatorname{cn}\xi\Big). \tag{119}$$

Case 8. If $a_0 = 1$, $a_2 = 2m^2 - 1$, $a_4 = (m^2 - 1)m^2$, then

$$y = \mathrm{sd}\,\xi, \quad b_0 = 0,$$

$$b_1 = \pm\sqrt{\frac{-\alpha + 2\beta + (2m^2 - 1)k^2c}{2k^2c}}, \tag{120}$$

$$c = \frac{2(1 - 2m^2)\beta \pm \sqrt{\alpha^2 - 16\beta^2m^2(1 - m^2)}}{k^2},$$

with

$$\alpha^2 - 16\beta^2m^2(1 - m^2) \geq 0, \quad 0 < m < 1, \tag{121}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{28} = 2\tan^{-1}\Big(\pm\sqrt{\frac{2k^2c}{-\alpha + 2\beta + (2m^2 - 1)k^2c}}\mathrm{ds}\,\xi\Big). \tag{122}$$

Case 9. If $a_0 = (m^2 - 1)m^2$, $a_2 = 2m^2 - 1$, $a_4 = 1$, then

$$y = \mathrm{ds}\,\xi, \quad b_0 = 0,$$

$$b_1 = \pm\sqrt{\frac{-\alpha + 2\beta + (2m^2 - 1)k^2c}{2(m^2 - 1)m^2k^2c}}, \tag{123}$$

$$c = \frac{2(1 - 2m^2)\beta \pm \sqrt{\alpha^2 - 16\beta^2m^2(1 - m^2)}}{k^2},$$

with

$$\alpha^2 - 16\beta^2m^2(1 - m^2) \geq 0, \quad 0 < m < 1, \tag{124}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{29} = 2\tan^{-1}\Big(\pm\sqrt{\frac{2(m^2 - 1)m^2k^2c}{-\alpha + 2\beta + (2m^2 - 1)k^2c}}\mathrm{sd}\,\xi\Big). \tag{125}$$

Case 10. If $a_0 = 1$, $a_2 = -(1 + m^2)$, $a_4 = m^2$, then

$$y = \mathrm{sn}\,\xi, \quad b_0 = 0,$$

$$b_1 = \pm\sqrt{\frac{-\alpha + 2\beta - (1 + m^2)k^2c}{2k^2c}}, \tag{126}$$

$$c = \frac{2(1 + m^2)\beta \pm \sqrt{(1 - m^2)^2\alpha^2 + 16\beta^2m^2}}{(1 - m^2)^2k^2},$$

with

$$\frac{2\beta}{k^2c} > (1 + m^2), \quad 0 < m < 1, \tag{127}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{30} = 2\tan^{-1}\Big(\pm\sqrt{\frac{2k^2c}{-\alpha + 2\beta - (1 + m^2)k^2c}}\mathrm{ns}\,\xi\Big). \tag{128}$$

Case 11. If $a_0 = 1$, $a_2 = -(1 + m^2)$, $a_4 = m^2$, then

$$y = \mathrm{cd}\,\xi, \quad b_0 = 0,$$

$$b_1 = \pm\sqrt{\frac{-\alpha + 2\beta - (1 + m^2)k^2c}{2k^2c}}, \tag{129}$$

$$c = \frac{2(1 + m^2)\beta \pm \sqrt{(1 - m^2)^2\alpha^2 + 16\beta^2m^2}}{(1 - m^2)^2k^2},$$

with

$$\frac{2\beta}{k^2c} > (1 + m^2), \quad 0 < m < 1, \tag{130}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{31} = 2\tan^{-1}\Big(\pm\sqrt{\frac{2k^2c}{-\alpha + 2\beta - (1 + m^2)k^2c}}\mathrm{dc}\,\xi\Big). \tag{131}$$

Case 12. If $a_0 = m^2$, $a_2 = -(1 + m^2)$, $a_4 = 1$, then

$$y = \mathrm{ns}\,\xi, \quad b_0 = 0,$$

$$b_1 = \pm\sqrt{\frac{-\alpha + 2\beta - (1 + m^2)k^2c}{2m^2k^2c}}, \tag{132}$$

$$c = \frac{2(1 + m^2)\beta \pm \sqrt{(1 - m^2)^2\alpha^2 + 16\beta^2m^2}}{(1 - m^2)^2k^2},$$

with

$$\frac{2\beta}{k^2c} > (1 + m^2), \quad 0 < m < 1, \tag{133}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{32} = 2\tan^{-1}\Big(\pm\sqrt{\frac{2m^2k^2c}{-\alpha + 2\beta - (1 + m^2)k^2c}}\mathrm{sn}\,\xi\Big). \tag{134}$$

Case 13. If $a_0 = m^2$, $a_2 = -(1+m^2)$, $a_4 = 1$, then

$$y = \mathrm{dc}\,\xi, \quad b_0 = 0,$$

$$b_1 = \pm\sqrt{\frac{-\alpha+2\beta-(1+m^2)k^2c}{2m^2k^2c}}, \tag{135}$$

$$c = \frac{2(1+m^2)\beta \pm \sqrt{(1-m^2)^2\alpha^2+16\beta^2m^2}}{(1-m^2)^2k^2},$$

with

$$\frac{2\beta}{k^2c} > (1+m^2), \quad 0 < m < 1, \tag{136}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{33} = 2\tan^{-1}\Big(\pm\sqrt{\frac{2m^2k^2c}{-\alpha+2\beta-(1+m^2)k^2c}}\mathrm{cd}\,\xi\Big). \tag{137}$$

Case 14. If $a_0 = -(1-m^2)$, $a_2 = 2-m^2$, $a_4 = -1$, then

$$y = \mathrm{dn}\xi, \quad b_0 = 0,$$

$$b_1 = \pm\sqrt{-\frac{-\alpha+2\beta+(2-m^2)k^2c}{2(1-m^2)k^2c}}, \tag{138}$$

$$c = \frac{2(m^2-2)\beta \pm \sqrt{m^2\alpha^2+16\beta^2(1-m^2)}}{m^4k^2},$$

with

$$\frac{2\beta}{k^2c} + (2-m^2) < 0, \quad 0 < m < 1, \tag{139}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{34} = 2\tan^{-1}\Big(\pm\sqrt{-\frac{2(1-m^2)k^2c}{-\alpha+2\beta+(2-m^2)k^2c}}\mathrm{nd}\,\xi\Big). \tag{140}$$

Case 15. If $a_0 = -1$, $a_2 = 2-m^2$, $a_4 = -(1-m^2)$, then

$$y = \mathrm{nd}\,\xi, \quad b_0 = 0,$$

$$b_1 = \pm\sqrt{-\frac{-\alpha+2\beta+(2-m^2)k^2c}{2k^2c}}, \tag{141}$$

$$c = \frac{2(m^2-2)\beta \pm \sqrt{m^2\alpha^2+16\beta^2(1-m^2)}}{m^4k^2},$$

with

$$\frac{2\beta}{k^2c} + (2-m^2) < 0, \quad 0 < m < 1, \tag{142}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{35} = 2\tan^{-1}\Big(\pm\sqrt{-\frac{2k^2c}{-\alpha+2\beta+(2-m^2)k^2c}}\mathrm{dn}\,\xi\Big). \tag{143}$$

Case 16. If $a_0 = 1$, $a_2 = 2-m^2$, $a_4 = 1-m^2$, then

$$y = \mathrm{sc}\,\xi, \quad b_0 = 0,$$

$$b_1 = \pm\sqrt{\frac{-\alpha+2\beta+(2-m^2)k^2c}{2k^2c}}, \tag{144}$$

$$c = \frac{2(m^2-2)\beta \pm \sqrt{m^2\alpha^2+16\beta^2(1-m^2)}}{m^4k^2},$$

with

$$\frac{2\beta}{k^2c} + (2-m^2) > 0, \quad 0 < m < 1, \tag{145}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{36} = 2\tan^{-1}\Big(\pm\sqrt{\frac{2k^2c}{-\alpha+2\beta+(2-m^2)k^2c}}\mathrm{cs}\,\xi\Big). \tag{146}$$

Case 17. If $a_0 = 1-m^2$, $a_2 = 2-m^2$, $a_4 = 1$, then

$$y = \mathrm{cs}\xi, \quad b_0 = 0,$$

$$b_1 = \pm\sqrt{\frac{-\alpha+2\beta+(2-m^2)k^2c}{2(1-m^2)k^2c}}, \tag{147}$$

$$c = \frac{2(m^2-2)\beta \pm \sqrt{m^2\alpha^2+16\beta^2(1-m^2)}}{m^4k^2},$$

with

$$\frac{2\beta}{k^2c} + (2-m^2) > 0, \quad 0 < m < 1, \tag{148}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{37} = 2\tan^{-1}\Big(\pm\sqrt{\frac{2(1-m^2)k^2c}{-\alpha+2\beta+(2-m^2)k^2c}}\mathrm{sc}\,\xi\Big). \tag{149}$$

Case 18. If $a_0 = \frac{1-m^2}{4}$, $a_2 = \frac{1+m^2}{2}$, $a_4 = \frac{1-m^2}{4}$, then

$$y = \frac{\mathrm{cn}\,\xi}{1 \pm \mathrm{sn}\,\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{2(-\alpha+2\beta)+(1+m^2)k^2c}{(1-m^2)k^2c}}, \tag{150}$$
$$c = \frac{-(1+m^2)\beta \pm \sqrt{m^2\alpha^2+\beta^2(1-m^2)^2}}{m^2k^2},$$

with

$$\frac{4\beta}{k^2c} + (1+m^2) > 0, \quad 0 < m < 1, \tag{151}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{38} = 2\tan^{-1}\Big(\pm\sqrt{\frac{(1-m^2)k^2c}{2(-\alpha+2\beta)+(1+m^2)k^2c}} \cdot \frac{1\pm\mathrm{sn}\,\xi}{\mathrm{cn}\,\xi}\Big). \tag{152}$$

Case 19. If $a_0 = -\frac{1-m^2}{4}$, $a_2 = -\frac{1+m^2}{2}$, $a_4 = \frac{1-m^2}{4}$, then

$$y = \frac{\mathrm{dn}\,\xi}{1 \pm m\mathrm{sn}\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{-\frac{2(-\alpha+2\beta)+(1+m^2)k^2c}{(1-m^2)k^2c}}, \tag{153}$$
$$c = \frac{-(1+m^2)\beta \pm \sqrt{m^2\alpha^2+\beta^2(1-m^2)^2}}{m^2k^2},$$

with

$$\frac{4\beta}{k^2c} + (1+m^2) < 0, \quad 0 < m < 1, \tag{154}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{39} = 2\tan^{-1}\Big(\pm\sqrt{-\frac{(1-m^2)k^2c}{2(-\alpha+2\beta)+(1+m^2)k^2c}} \cdot \frac{1\pm m\mathrm{sn}\,\xi}{\mathrm{dn}\,\xi}\Big). \tag{155}$$

Case 20. If $a_0 = \frac{m^2}{4}$, $a_2 = -\frac{2-m^2}{2}$, $a_4 = \frac{m^2}{4}$, then

$$y = \frac{m\mathrm{sn}\,\xi}{1 \pm \mathrm{dn}\,\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{2(-\alpha+2\beta)-(2-m^2)k^2c}{m^2k^2c}}, \tag{156}$$
$$c = \frac{(2-m^2)\beta \pm \sqrt{(1-m^2)\alpha^2+\beta^2m^4}}{(1-m^2)k^2},$$

with

$$\frac{4\beta}{k^2c} > (2-m^2), \quad 0 < m < 1, \tag{157}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{40} = 2\tan^{-1}\Big(\pm\sqrt{\frac{m^2k^2c}{2(-\alpha+2\beta)-(2-m^2)k^2c}} \cdot \frac{1\pm\mathrm{dn}\,\xi}{m\mathrm{sn}\,\xi}\Big). \tag{158}$$

Case 21. If $a_0 = \frac{1}{4}$, $a_2 = \frac{1-2m^2}{2}$, $a_4 = \frac{1}{4}$, then

$$y = \frac{\mathrm{sn}\,\xi}{1 \pm \mathrm{cn}\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{2(-\alpha+2\beta)+(1-2m^2)k^2c}{k^2c}}, \tag{159}$$
$$c = \frac{(2m^2-1)\beta \pm \sqrt{m^2(m^2-1)\alpha^2+\beta^2}}{m^2(m^2-1)k^2},$$

with

$$m^2(m^2-1)\alpha^2+\beta^2 > 0, \quad 0 < m < 1, \tag{160}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{41} = 2\tan^{-1}\Big(\pm\sqrt{\frac{k^2c}{2(-\alpha+2\beta)+(1-2m^2)k^2c}} \cdot \frac{1\pm\mathrm{cn}\,\xi}{\mathrm{sn}\,\xi}\Big). \tag{161}$$

Case 22. If $a_0 = \frac{1}{4}$, $a_2 = -\frac{2-m^2}{2}$, $a_4 = \frac{m^4}{4}$, then

$$y = \frac{\mathrm{sn}\,\xi}{1 \pm \mathrm{dn}\xi}, \quad b_0 = 0,$$
$$b_1 = \pm\sqrt{\frac{2(-\alpha+2\beta)-(2-m^2)k^2c}{k^2c}}, \tag{162}$$
$$c = \frac{(2-m^2)\beta \pm \sqrt{(1-m^2)\alpha^2+\beta^2m^4}}{(1-m^2)k^2},$$

with

$$\frac{4\beta}{k^2c} > (2-m^2), \quad 0 < m < 1, \tag{163}$$

where k is an arbitrary constant. So the solution to the DSG equation (2) is

$$u_{42} = 2\tan^{-1}\Big(\pm\sqrt{\frac{k^2c}{2(-\alpha+2\beta)-(2-m^2)k^2c}} \cdot \frac{1\pm \mathrm{dn}\,\xi}{\mathrm{sn}\,\xi}\Big). \tag{164}$$

Remark. Most of the solutions from u_{23} to u_{42} in terms of Jacobian elliptic functions have not been given in the literature.

4. Conclusion

In this paper, two transformations are introduced to solve the double sine-Gordon equation by using knowledge of the elliptic equation and Jacobian elliptic functions. It is shown that different transformations can be used to obtain more kinds of solutions to the double sine-Gordon equation. In this paper, we have found some solutions that have not been reported in the literature. It is shown that different transformations play different roles in obtaining exact solutions, some transformations may not work for a specific parameter of the DSG equation. Of course, still more efforts are needed to explore what kinds of transformations are more suitable to solve the sine-Gordon equation. Because different transformations result in different partial balances for the sine-Gordon equation, which will lead to different expansion truncations in the elliptic equation expansion method. Finally, these will result in different solutions of the sine-Gordon equation.

Acknowledgement

Many thanks are due to helpful suggestions from anonymous referees and support from National Natural Science Foundation of China (No. 40305006).

[1] E. Goldobin, A. Sterck, and D. Koelle, Phys. Rev. E **63**, 03111 (2001).
[2] K. M. Leung, D. L. Mills, P. S. Riseborough, and S. E. Trullinger, Phys. Rev. B **27**, 4017 (1983).
[3] M. Salerno and N. R. Quintero, Phys. Rev. E **65**, 025602 (2002).
[4] S. Y. Lou and G. J. Ni, Phys. Lett. A **140**, 33 (1989).
[5] V. A. Gani and A. E. Kudryavtsev, Phys. Rev. E **60**, 3305 (1999).
[6] R. K. Bullough, P. J. Caudrey, and H. M. Gibbs, in: Solitons (Eds. R. K. Bullough and P. J. Caudrey), Springer, Berlin 1980.
[7] M. Panigrahi and P. C. Dash, Phys. Lett. A **321**, 330 (2004).
[8] J. S. Yang and S. Y. Lou, Chin. Phys. Lett. **21**, 608 (2004).
[9] E. G. Fan and J. Zhang, Phys. Lett. A **305**, 383 (2002).
[10] Y. Z. Peng, Phys. Lett. A **314**, 402 (2003).
[11] Sirendaoreji and J. Sun, Phys. Lett. A **298**, 133 (2002).
[12] K. W. Chow, Wave Motion **35**, 71 (2002).
[13] Z. T. Fu, S. D. Liu, and S. K. Liu, Commun. Theor. Phys. **39**, 531 (2003).
[14] Z. T. Fu, S. K. Liu, and S. D. Liu, Commun. Theor. Phys. **40**, 285 (2003).
[15] Z. T. Fu, Z. Chen, S. K. Liu, and S. D. Liu, Commun. Theor. Phys. **41**, 675 (2004).
[16] Z. T. Fu, Z. Chen, S. D. Liu, and S. K. Liu, Commun. Theor. Phys. **42**, 43 (2004).
[17] Z. T. Fu, S. K. Liu, and S. D. Liu, Commun. Theor. Phys. **42**, 343 (2004).
[18] Z. T. Fu, S. K. Liu, S. D. Liu, and Q. Zhao, Phys. Lett. A **290**, 72 (2001).
[19] S. K. Liu, Z. T. Fu, S. D. Liu, and Q. Zhao, Phys. Lett. A **289**, 69 (2001).
[20] S. K. Liu and S. D. Liu, Nonlinear Equations in Physics, Peking University Press, Beijing 2000.
[21] F. Bowman, Introduction to Elliptic Functions with Applications, Universities, London 1959.
[22] V. Prasolov and Y. Solovyev, Elliptic Functions and Elliptic Integrals, American Mathematical Society, Providence, R.I. 1997.
[23] Z. X. Wang and D. R. Guo, Special Functions, World Scientific Publishing, Singapore 1989.